



# Domination with exponential decay

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## ABSTRACT

Let  $G$  be a graph and  $S \subseteq V(G)$ . For each vertex  $u \in S$  and for each  $v \in V(G) - S$ , we define  $\bar{d}(u, v) = \bar{d}(v, u)$  to be the length of a shortest path in  $\langle V(G) - (S - \{u\}) \rangle$  if such a path exists, and  $\infty$  otherwise. Let  $v \in V(G)$ . We define  $w_S(v) = \sum_{u \in S} \frac{1}{2^{\bar{d}(u, v) - 1}}$  if  $v \notin S$ , and  $w_S(v) = 2$  if  $v \in S$ . If, for each  $v \in V(G)$ , we have  $w_S(v) \geq 1$ , then  $S$  is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number*,  $\gamma_e(G)$ . In this paper, we prove: (i) that if  $G$  is a connected graph of diameter  $d$ , then  $\gamma_e(G) \geq (d+2)/4$ , and, (ii) that if  $G$  is a connected graph of order  $n$ , then  $\gamma_e(G) \leq \frac{2}{3}(n+2)$ .

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## 1. Introduction

Throughout this article, for terms that are not explained here, we refer the reader to the book by Chartrand and Lesniak [1].

The well-known concept of domination in graphs is a good tool for analyzing situations that can be modeled by networks in which a vertex can exert influence on, or dominate, all vertices in its immediate neighborhood. In some real world situations, a vertex can influence not only the vertices within its immediate neighborhood, but also all vertices within a given distance. This kind of situation is captured by distance domination. However, no framework for situations in which the influence of a vertex extends beyond its neighborhood but decreases with distance has been put forward yet. In this paper, we propose such a framework. In our model, the ‘dominating power’ of a vertex decreases exponentially, by the factor  $\frac{1}{2}$ , with distance. Hence a vertex  $v$  can be dominated by a neighbor of  $v$  or by a number of vertices that are not too far from  $v$ . Such a model could be used, for example, for the analysis of dissemination of information in social networks, where the impact of the information decreases every time it is passed on.

We first recall the idea of a classical dominating set. Let  $G$  be a graph and  $S \subseteq V(G)$ . Then  $S$  is a *dominating set* if every vertex in  $V(G) - S$  is adjacent to at least one member of  $S$ . The minimum cardinality of a dominating set is the *domination number*,  $\gamma(G)$ . Domination in graphs has received a considerable amount of attention (see, for example, the books [6,5] by Haynes, Hedetniemi, and Slater).

Domination at a distance has also been studied. A set  $S \subseteq V(G)$  is a *k-dominating set* if every vertex of  $G$  is within distance  $k$  of some member of  $S$ . The minimum cardinality of a *k-dominating set* is the *k-domination number*,  $\gamma_k(G)$ ; for more details, the reader is referred to the survey article [8] by Henning.

Other variations on domination at a distance have received some attention. Slater [10] studied the situation in which for each vertex  $v$  there is a number  $f(v)$ , and we require that each  $v$  be within distance  $f(v)$  of some member of  $S$ . Erwin [4] studied the converse situation: Each vertex  $v$  is assigned a nonnegative integer  $f(v)$ ; if  $f(v) > 0$ , then  $v$  dominates every vertex within distance  $f(v)$ .

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In this article, we introduce a variation of distance domination in which, as described in the motivation already given, the ‘dominating power’ radiating from a vertex declines exponentially with distance. Let  $G$  be a graph and  $S \subseteq V(G)$ . We denote by  $\langle S \rangle$  the subgraph of  $G$  induced by  $S$ . For each vertex  $u \in S$  and for each  $v \in V(G) - S$ , we define  $\bar{d}(u, v) = \bar{d}(v, u)$  to be the length of a shortest path in  $\langle V(G) - (S - \{u\}) \rangle$  if such a path exists, and  $\infty$  otherwise. Let  $v \in V(G)$ . We define

$$w_S(v) = \begin{cases} \sum_{u \in S} \frac{1}{2^{\bar{d}(u,v)-1}}, & \text{if } v \notin S, \\ 2, & \text{if } v \in S. \end{cases}$$

We refer to  $w_S(v)$  as the *weight* of  $S$  at  $v$  (note that we define  $w_S(v) = 2$  if  $v \in S$  since then  $v$  contributes  $w_S(v)/2^d$  to every vertex it exponentially dominates at distance  $d$ ). If, for each  $v \in V(G)$ , we have  $w_S(v) \geq 1$ , then  $S$  is an *exponential dominating set*. The smallest cardinality of an exponential dominating set is the *exponential domination number*,  $\gamma_e(G)$ , and such a set is a *minimum exponential dominating set*, or  $\gamma_e$ -set for short. If  $u \in S$  and  $v \in V(G) - S$  and  $\frac{1}{2^{\bar{d}(u,v)-1}} > 0$ , then we say that  $u$  *exponentially dominates*  $v$ . Note that if  $S$  is an exponential dominating set, then every vertex of  $V(G) - S$  is exponentially dominated, but the converse is not true.

Finally, we note that sets whose influence diminishes exponentially with distance have been studied, though not in the context of domination. Daugherty et al. [2,3] define the *influence number* of a graph  $G$  as  $\eta(G) = \max_{S \subseteq V(G)} \eta(S)$ , where

$$\eta(S) = \sum_{u \in S} \frac{1}{2^{d(u,S)}},$$

and the *total influence number* of  $G$  as  $\eta_T(G) = \max_{S \subseteq V(G)} \eta_T(S)$ , where

$$\eta_T(S) = \sum_{v \in S} \sum_{u \in S} \frac{1}{2^{d(u,v)}}.$$

## 2. Elementary results

We note first of all that, for every graph  $G$ ,  $\gamma_e(G) \leq \gamma(G)$ . Also,  $\gamma_e(G) = 1$  if and only if  $\gamma(G) = 1$ .

Before giving our main results, we now, for the purpose of further acquainting the reader with the parameter, give some examples of its calculation. Following [1], we denote by  $P_n$  the path of order  $n$ . Also, if  $u, v$  are vertices of  $G$ , then  $[u, v]$  will denote the set of all vertices of  $G$  that lie on at least one  $u - v$  geodesic.

**Proposition 1.** For every positive integer  $n$ ,

$$\gamma_e(P_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

**Proof.** Let  $S$  be a minimum exponential dominating set of  $P_n$ . We note that each vertex of  $P_n - S$  is exponentially dominated by at most two vertices of  $S$ . Let  $u, v$  be vertices of  $S$  on  $P_n$  such that  $[u, v] \cap S = \{u, v\}$  and  $u = x_0, x_1, \dots, x_k = v$  the  $u - v$  path in  $P_n$ . The vertex  $u$  contributes  $1/2$  to  $w_S(x_2)$ , so  $v$  must contribute at least  $1/2$  to  $w_S(x_2)$ . This implies that  $d(u, v) \leq 4$ , which in turn implies that  $n \leq 4\gamma_e(P_n) - 1$ . To see that  $\gamma_e(P_n) \leq \lceil (n+1)/4 \rceil$ , we note that it is easy to construct an exponential dominating set  $S$  with  $|S| = \lceil (n+1)/4 \rceil$ .  $\square$

Let  $C_n$  denote the cycle of order  $n$ . The following result can be proved using a similar argument:

**Proposition 2.** For every integer  $n \geq 3$ ,

$$\gamma_e(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \neq 4. \end{cases}$$

## 3. A lower bound on the exponential domination number

Before we establish a lower bound on the exponential domination number, we shall need two lemmas. For the first lemma, we shall need the following theorem due to Helly [7]:

**Theorem 3 (Helly's Condition).** Given  $t$  convex sets in  $\mathbb{R}^d$  where  $d < t$ , if every  $d+1$  of them intersect at a common point, then they all intersect at a common point.

**Lemma 4.** Let  $G$  be a connected graph,  $P$  a geodesic in  $G$ , and  $x \in V(G)$ . Then there is a vertex  $y \in V(P)$  such that for all  $z \in V(P)$ ,  $d(y, z) \leq d(x, z)$ .

**Proof.** For each vertex  $z \in V(P)$ , let  $N_x(z) = \{v \in V(P) : d(z, v) \leq d(z, x)\}$ . Then each  $N_x(z)$  is a convex subset of  $\mathbb{Z}$ . Consider two vertices  $z_1, z_2 \in V(P)$ . Then  $d(z_1, x) + d(x, z_2) \geq d(z_1, z_2)$ , which implies that  $N_x(z_1) \cap N_x(z_2) \neq \emptyset$ . Hence,  $\{N_x(z) : z \in V(P)\}$  is a family of sets that satisfies Theorem 3; consequently, there exists  $y \in \bigcap_{z \in V(P)} N_x(z)$ .  $\square$

For the second lemma, we need some more terminology. Let  $G$  be a graph and  $S$  a set of vertices of  $G$ . For  $v \in V(G)$ , define

$$w^*(v) = \sum_{s \in S} \frac{1}{2^{d(v,s)-1}}.$$

A *porous exponential dominating set* (or *p-exponential dominating set*) of  $G$  is a set  $S \subseteq V(G)$  with  $w^*(v) \geq 1$  for all  $v \in V(G)$  (we have called it porous since, unlike ‘regular’ exponential domination, the dominating power of a vertex of  $S$  is allowed to ‘seep through’ another member of  $S$ ). The minimum cardinality of a porous exponential dominating set is the *p-exponential domination number*, denoted by  $\gamma_e^*(G)$ . We sketch the proof of the following theorem and leave the reader to insert the details.

**Lemma 5.** For every positive integer  $n$ ,

$$\gamma_e^*(P_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

**Proof.** Since  $\gamma_e^*(P_n) \leq \gamma_e(P_n)$ , we have  $\gamma_e^*(P_n) \leq \lceil \frac{n+1}{4} \rceil$ . It remains to show that every  $p$ -exponential dominating set of  $P_n$  has at least  $\lceil \frac{n+1}{4} \rceil$  vertices.

Let  $V(P_n) = \{0, 1, 2, \dots, n-1\}$ . Let  $S = \{a_1, a_2, \dots, a_k\}$  be a  $p$ -exponential dominating set of  $P_n$  with  $a_1 < a_2 < \dots < a_k$ . Let  $\ell_i = a_{i+1} - a_i$  for  $i = 1, 2, \dots, k-1$ . We first note that  $\ell_i \leq 5$  for  $i = 1, 2, \dots, k-1$ . This follows from the fact that  $\ell_i \geq 6$  would imply that  $S \subseteq \{1, 2, \dots, a_i\} \cup \{a_i + 6, a_i + 7, \dots, n\} =: S'$ . But it is easy to verify that  $S'$  (and thus  $S$ ) do not exponentially dominate  $a_i + 3$ , a contradiction. Similarly, we show that  $\ell_i = 5$  implies that  $\min(\ell_{i-1}, \ell_{i+1}) \leq 2$ . It is easy to show that these two statements imply  $\sum_{i=1}^k \ell_i \leq 4(k-1)$ . Since we can assume that the set  $S$  does not contain an end vertex, but since the end vertices need to be exponentially dominated, we have  $a_1 = 1$  and  $a_k = n-1$ . This yields

$$n-3 = \sum_{i=1}^k \ell_i \leq 4(k-1),$$

and thus  $|S| = k \geq (n+1)/4$ , as desired.  $\square$

We are now ready to state the lower bound.

**Theorem 6.** If  $G$  is a connected graph of diameter  $d$ , then

$$\gamma_e(G) \geq \left\lceil \frac{d+2}{4} \right\rceil.$$

**Proof.** Let  $S$  be an exponential dominating set of  $G$  and  $P : v_0, v_1, \dots, v_d$  a diameter-length geodesic in  $G$ . We claim that  $|S| \geq (d+2)/4$ . Suppose, to the contrary, that this is not the case. By Lemma 4, for each vertex  $v \in S$ , there is a vertex  $f(v) \in V(P)$  such that for all  $z \in V(P)$ ,  $d(f(v), z) \leq d(v, z)$ . By Lemma 5,  $S' = \{f(v) : v \in S\}$  is a multiset (if it were a set but not a multiset, its cardinality would be smaller than the  $p$ -exponential domination number of  $P_n$ , contradicting Lemma 5). Also, the set  $S'$  is clearly a  $p$ -exponential dominating multiset of  $P$ .

We now describe how to reduce  $S'$  to a  $p$ -exponential dominating set of  $P$ . Suppose  $v_i$  has multiplicity  $m \geq 2$  in  $S'$ . Since  $|S'| < (d+2)/4$ , for every vertex of  $S'$ , there are roughly three vertices of  $P$  that are not in  $S'$ . Let  $k = \lfloor m/2 \rfloor$ , and let  $i_1, i_2, \dots, i_k$  be the largest integers less than  $i$  such that for each  $t \in [1, k]$ , the vertex  $v_{i_t}$  is not in  $S'$ . Similarly, let  $j_1, j_2, \dots, j_k$  be the smallest integers greater than  $i$  such that for each  $t \in [1, k]$ , the vertex  $v_{j_t}$  is not in  $S'$  (if there are not enough ‘empty’ vertices above or below  $v_i$ , then the leftover vertices can be added below or above  $v_i$ , as needed). Remove the  $m$  instances of  $v_i$  from  $S'$ , and replace them with the vertices  $v_{i_1}, v_{i_2}, \dots, v_{i_k}, v_{j_1}, v_{j_2}, \dots, v_{j_k}$ . Finally, if  $m$  is odd, add the vertex  $v_i$  to  $S'$ . Then  $S'$  is a  $p$ -exponential dominating multiset in which one fewer vertex has multiplicity greater than 1. By repeating this procedure for each vertex of  $S'$  with multiplicity greater than 1, we produce a  $p$ -exponential dominating set  $S''$  with  $|S''| = |S'| < (d+2)/4$ , contradicting Lemma 5.  $\square$

Note that, by Proposition 1, this bound is sharp. Moreover, this bound is sharp for infinitely many graphs that are not paths (simply take a path  $P$  of diameter  $d$ , construct a minimum exponential dominating set  $S$  of  $P$ , then add endvertices adjacent to the vertices of  $S$ ).

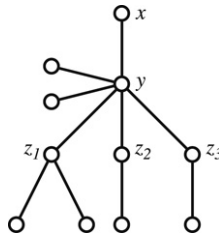


Fig. 1. A  $3_{2,1,1}^2$ -branch.

#### 4. An upper bound on the exponential domination number

Let  $G$  be a connected graph. If  $v$  is a vertex of  $G$ , then  $v$  is *peripheral* if, amongst all the vertices of  $G$ ,  $v$  has maximum eccentricity.

**Theorem 7.** If  $G$  is a connected graph of order  $n$ , then

$$\gamma_e(G) \leq \frac{2}{5}(n+2). \quad (1)$$

**Proof.** Let  $G$  be a connected graph of order  $n$  and  $T$  a spanning tree of  $G$ . Then  $\gamma_e(G) \leq \gamma_e(T)$ . It therefore suffices to prove the result for the case when  $G$  is a tree.

Suppose that the theorem is not true. Then there exists a tree whose exponential domination number is not bounded above by the right-hand side of Eq. (1). Amongst all such trees, let  $T$  be one of minimum order  $n$ . It can be verified by hand that every tree of order at most 10 satisfies Eq. (1), so  $n \geq 11$ . Let  $r$  be a peripheral vertex of  $T$  and consider  $T$  as a tree rooted at  $r$ . Given a vertex  $x$  of  $T$ , we define the *maximal subtree at  $x$* ,  $T_x$ , to consist of  $x$  and all its descendants (i.e., all those vertices  $q$  of  $T$  for which  $x$  lies on the unique  $q-r$  path). Note that  $T = T_r$ . Given a maximal subtree  $T_x$ , we define the *depth* of  $T_x$  to be the eccentricity in  $T_x$  of  $x$ . Thus the depth of  $T$  is the diameter of  $T$ .

If  $T$  has depth 0 or 1, then  $n < 11$ , a contradiction. If  $T$  has depth 2, then  $T$  is the star  $K_{1,n-1}$  and  $\gamma_e(T) = 1 \leq \frac{2}{5}(2+2) \leq \frac{2}{5}(n+2)$ , a further contradiction.

Assume now that  $T$  has depth 3. Since  $r$  is peripheral,  $T$  consists of two adjacent vertices  $u, v$  and two sets,  $\{u_1, \dots, u_y\}$  and  $\{v_1, \dots, v_z\}$ , of vertices such that each  $u_i$  is adjacent to  $u$  and each  $v_i$  is adjacent to  $v$ . Then  $\gamma_e(T) = 2 \leq \frac{2}{5}(n+2)$ , a contradiction.

It follows from this discussion that  $T$  contains vertices  $p$  and  $x$  such that  $T_p$  has depth 4,  $T_x$  has depth 3, and  $x$  is a child of  $p$ . Every maximal nontrivial path beginning at  $x$  and not including  $p$  has length 1, 2, or 3. We shall now need more terminology. Let  $y \neq p$  be a vertex adjacent to  $x$ . Then the vertex  $x$ , the edge  $xy$ , and the tree  $T_y$  will be collectively called a *branch at  $x$* ; a branch of depth  $k$  will be called a  $k$ -branch. Let  $B$  be a branch at  $x$ . If  $B$  has depth 1, then  $B$  is  $K_2$ , and will be called a 1-branch. If  $B$  has depth 2, then  $B$  consists of the vertex  $x$ , the vertex  $y$ , and  $k$  endvertices adjacent to  $y$ . In this case, we shall call  $B$  a  $2^k$ -branch. If  $B$  has depth 3, then  $B$  consists of  $x$  and  $y$ , together with (i)  $k_0$  endvertices adjacent to  $y$ , and, (ii) a collection of subtrees of depth 2 rooted at  $y$ . Each of these subtrees of depth 2 rooted at  $y$  consists of  $y$ , a vertex  $z$  adjacent to  $y$ , and a number of endvertices adjacent to  $z$ . Let  $\{z_i : i \in \{1, 2, \dots, \ell\}\}$  be the set of children of  $y$  that have degree at least 2. If each  $z_i$  is adjacent to  $k_i$  endvertices, then we call  $B$  a  $3_{k_1, k_2, \dots, k_\ell}^{k_0}$ -branch. As an example, a  $3_{2,1,1}^2$ -branch is shown in Fig. 1.

We now make some observations:

- **$x$  has no  $2^k$ -branch for any  $k \geq 2$ :** For suppose that  $x$  has such a branch. Let  $T' = T - T_y$  and let  $S'$  be an exponential dominating set of  $T'$ . By the minimality of  $T$ ,  $|S'| \leq \frac{2}{5}(n - (k+1) + 2)$ . Furthermore,  $S' \cup \{y\}$  is an exponential dominating set of  $T$ . Hence  $\gamma_e(T) \leq \frac{2}{5}(n - 3 + 2) + 1 \leq \frac{2}{5}(n+2)$ , contradicting our choice of  $T$ . Since every 2-branch is a  $2^1$ -branch, we shall henceforth call a  $2^1$ -branch simply a 2-branch.
- **If  $x$  has a  $3_{k_1, k_2, \dots, k_\ell}^{k_0}$ -branch, then  $k_1 = k_2 = \dots = k_\ell = 1$ .** Remove a branch at  $y$  that has at least two endvertices; then the result follows immediately from the preceding argument. For this reason, we henceforth simplify our terminology: a  $3_{1,1,\dots,1}^{k_0}$ -branch (there are  $\ell$  1's) will be called a  $3_\ell^{k_0}$ -branch.
- **There is no  $3_1^k$ -branch at  $x$ :** Suppose that there is a  $3_1^k$ -branch. If  $k = 0$ , then consider the tree  $T' = T - T_y$ . The vertex  $z$  dominates all three vertices of  $T_y$ ; hence, as previously,  $\gamma_e(T) \leq \frac{2}{5}(n - 3 + 2) + 1$ , contradicting our choice of  $T$ . Thus  $k \geq 1$ . Let  $A$  be the set of endvertices that are adjacent to  $y$ . Consider the tree  $T' = T - (T_z \cup A)$  and let  $S'$  be a minimum exponential dominating set of  $T'$ . By the minimality of  $T$ ,  $|S'| \leq \frac{2}{5}(n - (k+2) + 2)$ . If  $y \in S'$ , then  $S = S' \cup \{z\}$  is a dominating set of  $T$  with  $|S| \leq \frac{2}{5}(n - k) + 1 = \frac{2}{5}(n - k + 5/2) \leq \frac{2}{5}(n + 3/2)$ , contradicting our choice of  $T$ . Suppose then that  $y \notin S'$  and let  $e$  be any endvertex adjacent to  $y$ . Since  $S'$  exponentially dominates  $y$ ,  $w_{S'}(y) \geq 1$ , so, in  $T$ ,  $w_{S'}(e) \geq 1/2$ . Therefore,  $S = S' \cup \{z\}$  is an exponential dominating set, which (as previously) produces a contradiction. It follows, therefore, that there is no  $3_1^k$ -branch at  $x$ .

- **There is no  $3_\ell^k$  branch at  $x$  for any  $\ell \geq 4$ :** Suppose that  $x$  has a  $3_\ell^k$  branch for some  $\ell \geq 4$ . Let  $H = \bigcup_{i=1}^\ell T_{z_i}$  and  $D = \{z_1, z_2, z_3\}$ . Let  $T' = T - H$ . By the minimality of  $T$ , there is an exponential dominating set  $S'$  of  $T'$  satisfying  $|S'| \leq \frac{2}{5}(n - 2\ell + 2)$ . We consider two cases, according to whether or not  $y \in S'$ . Suppose first that  $y \notin S'$ . Let  $S = S' \cup D$ . Since  $w_{S'}(y) \geq 1$ ,  $S$  is an exponential dominating set of  $T$ . Suppose then that  $y \in S'$ . Let  $S = (S' \cup D \cup \{z_4\}) - \{y\}$ . Notice that  $w_{S'}(y) = 2$  and  $w_S(y) \geq 4$ , implying that for all  $v \in V(T')$ ,  $w_S(v) \geq w_{S'}(v)$ . Furthermore,  $S$  exponentially dominates all the vertices of  $H$ , so  $S$  is an exponential dominating set of  $T$ . Thus, whether or not  $y \in S'$ , we have an exponential dominating set  $S$  satisfying  $|S| \leq |S'| + 3 \leq \frac{2}{5}(n - 2\ell + 2) + 3 = \frac{2}{5}(n - 2\ell + 19/2) \leq \frac{2}{5}(n + 3/2)$ , again producing a contradiction. Hence  $x$  does not have a  $3_\ell^k$  branch for any  $\ell \geq 4$ .
- **Every 3-branch at  $x$  is a  $3_3^k$ -branch:** Suppose that  $x$  has a  $3_2^k$ -branch. Let  $T' = T - T_y$ . Then, as in previous discussions, by choosing the two vertices of degree 2 in  $T_y$  and adding them to a minimum exponential dominating set  $S'$  of  $T'$ , we obtain an exponential dominating set  $S$  of  $T$  with cardinality at most  $\frac{2}{5}(n + 2)$ , a contradiction. It follows that  $x$  does not have a  $3_2^k$ -branch, and, consequently, every 3-branch at  $x$  is a  $3_3^k$ -branch.
- **Every 3-branch at  $x$  is a  $3_3^0$ -branch:** Consider a  $3_3^k$ -branch having  $k \geq 1$ . Let  $T' = T - T_y$  and let  $S'$  be a minimum exponential dominating set of  $T'$ . Let  $D = \{z_1, z_2, z_3\}$ . Then  $S = S' \cup D$  is an exponential dominating set of  $T$  having  $|S| \leq |S'| + 3 \leq \frac{2}{5}(n - (k + 7) + 2) + 3 \leq \frac{2}{5}(n + 3/2) < \frac{2}{5}(n + 2)$ , contradicting our choice of  $T$ .
- **There are no 1-branches at  $x$ :** Suppose, to the contrary, that there are  $a \geq 1$  1-branches at  $x$ , and let  $A$  be the set of endvertices adjacent to  $x$ . From our choice of  $x$ , there is a  $3_3^0$ -branch  $B$  at  $x$ . Let  $y$  be the vertex of degree 4 in  $B$  and let  $D$  be the set of vertices of degree 2 in  $B$ . Let  $S'$  be a minimum exponential dominating set of  $T' = T - (A \cup T_y)$ . Then  $S = S' \cup D$  is an exponential dominating set of  $T$  (if  $x \notin S'$ , then certainly  $w_{S'}(x) \geq 1$ , so  $w_{S'}(e) \geq 1/2$  for all  $e \in A$ ). As before,  $|S| \leq \frac{2}{5}(n + 2)$ , which is a contradiction.
- **The number of  $3_3^0$ -branches at  $x$  is at most 2:** Suppose, to the contrary, that there are three  $3_3^0$ -branches at  $x$ :  $B_1, B_2, B_3$ . In the branch  $B_i$ , let  $y_i$  be the vertex of degree 4 and  $D_i$  the set of vertices of degree 2. Let  $z$  be any vertex in  $D' = D_1 \cup D_2 \cup D_3$ , and let  $D = D' - \{z\}$ . Let  $S'$  be a minimum exponential dominating set of  $T' = T - (T_{y_1} \cup T_{y_2} \cup T_{y_3})$ . If  $x \notin S'$ , then  $w_{S'}(x) \geq 1$ , so  $S = S' \cup D$  exponentially dominates  $T$ . On the other hand, if  $x \in S'$ , consider the set  $S = S' \cup D - \{x\}$ . Notice that  $w_{S'}(x) = 2$ , while  $w_S(x) \geq 4$ , so  $S$  exponentially dominates  $T$ . Hence, in either case, there is an exponential dominating set  $S$  of  $T$  containing at most  $\frac{2}{5}(n - 21 + 2) + 8$  vertices, once again contradicting our choice of  $T$ .
- **The number of 2-branches at  $x$  is at most 2:** Suppose, to the contrary, that there are three 2-branches at  $x$ . Then there is either one  $3_3^0$ -branch or two  $3_3^0$ -branches. Suppose, first, that there is one  $3_3^0$ -branch. Let  $A$  be the set of 2-branches at  $x$ ; let  $B$  be the  $3_3^0$ -branch. Let  $A'$  consist of any two vertices of degree 2 in  $A$ ; let  $B'$  be the set of vertices of degree 2 in  $B$ . Let  $S'$  be a minimum exponential dominating set of  $T' = T - (A \cup B)$ . Recalling that  $p$  is the parent of  $x$ , we note that since  $w_{S'}(p) \geq 1$ ,  $p$  contributes at least  $1/2$  to  $x$  and at least  $1/8$  to each endvertex in  $A$  that is not adjacent to a vertex of  $A'$ ; the remainder of the weight at such an endvertex comes from the vertices of  $A' \cup B'$ . Therefore,  $S = S' \cup A' \cup B'$  is an exponential dominating set of  $T$ , and  $|S| \leq \frac{2}{5}(n - 14 + 2) + 5$ , which is a contradiction. It follows that there must be two  $3_3^0$ -branches at  $x$ . A similar argument again produces a contradiction.

It follows from the preceding observations that  $T_x$  consists of  $i$  2-branches and  $j$   $3_3^0$ -branches, where  $i \in \{0, 1, 2\}$  and  $j \in \{1, 2\}$ . We now consider two cases:

- Case 1:  $i = 0$ : Let  $S'$  be a minimum exponential dominating set of  $T' = T - T_x$  and  $Z$  the set of vertices of degree 2 in  $T_x$ . Then  $S = S' \cup Z$  exponentially dominates  $T$ . If  $j = 1$ , then  $|S| \leq \frac{2}{5}(n - 8 + 2) + 3$ , a contradiction. On the other hand, if  $j = 2$ , then  $|S| \leq \frac{2}{5}(n - 15 + 2) + 6$ , which is once again a contradiction.
- Case 2:  $i \geq 1$ : • **Subcase 2.1:**  $j = 2$ : Let  $y_1, y_2$  be the vertices of degree 4 in the  $3_3^0$ -branches at  $x$  and let  $D$  be the set of vertices of degree 2 in the two  $3_3^0$ -branches at  $x$ . Let  $A$  be the set of vertices of degree 2 in the 2-branches at  $x$ . Let  $T' = T - (T_{y_1} \cup T_{y_2} \cup [\bigcup_{a \in A} T_a])$  and let  $S'$  be a minimum exponential dominating set of  $T'$ . We consider two subcases. Suppose first that  $x \notin S'$ . Then  $w_{S'}(x) \geq 1$ , so  $S = (S' \cup D)$  exponentially dominates  $T$ , and  $|S| \leq \frac{2}{5}(n - (2i + 14) + 2) + 6$ , which is again a contradiction. Suppose then that  $x \in S'$ . Let  $a \in A$ . Then  $S = (S' \cup D \cup \{a\}) - \{x\}$  is an exponential dominating set of  $T$  (note that  $w_S(x) = 4 > 2 = w_{S'}(x)$ ), and  $|S| \leq \frac{2}{5}(n - (2i + 14) + 2) - 1 + 7$ , which is again a contradiction.
- **Subcase 2.2:**  $j = 1$ : If  $i = 1$ , then (by first removing  $T_x$ ) we can again easily construct an exponential dominating set for  $T$  containing at most  $\frac{2}{5}(n + 2)$  vertices, so it follows that  $i = 2$ . Let  $a_1, a_2$  be the two vertices of degree 2 in the two 2-branches. Let  $y$  be the vertex of degree 4 in the  $3_3^0$ -branch, and let  $z_1, z_2, z_3$  be the vertices of degree 2 in the 3-branch. Consider the tree  $T' = T - (T_{z_1} \cup T_{z_2} \cup T_{z_3} \cup T_{a_1} \cup T_{a_2})$ . Let  $S'$  be a minimum exponential dominating set of  $T'$ . Then, without loss of generality, we may assume that  $y \notin S'$ . We now consider two cases:
- **Subcase 2.2.1:**  $x \notin S'$ : Since  $y$  is exponentially dominated,  $w_{S'}(x) \geq 2$ . Therefore  $S = S' \cup \{a_1, z_1, z_2, z_3\}$  is an exponential dominating set of  $T$ , and  $|S| \leq \frac{2}{5}(n - 10 + 2) + 4$ , a contradiction.
  - **Subcase 2.2.2:**  $x \in S'$ : Let  $S = S' \cup \{a_1, a_2, z_1, z_2, z_3\} - \{x\}$ . Then  $w_S(x) \geq 7/2 > 2 = w_{S'}(x)$ , so  $S$  exponentially dominates  $T$ ; moreover,  $|S| \leq \frac{2}{5}(n - 10 + 2) - 1 + 5$ , which (for the final time) produces a contradiction.

Hence no such tree  $T$  exists and the upper bound has been established.  $\square$

Note that, in some situations, our upper bound is implied by the existing results. For example, it is well known [1] that if  $G$  is a graph of order  $n$  without isolated vertices, then  $\gamma(G) \leq n/2$ . Since  $\lfloor n/2 \rfloor \leq \lfloor \frac{2}{5}(n+2) \rfloor$  if and only if  $n \in [1, 9] \cup \{11, 13\}$ , for these values of  $n$ , our bound is implied by the existing bound on the domination number.

In a similar vein, McCuaig and Shepherd [9] have proved that if  $G$  is a graph of order  $n$  and minimum degree two, and  $G$  is not one of seven exceptional graphs, then  $\gamma(G) \leq \frac{2}{5}n$ . When  $G$  has minimum degree 2, McCuaig and Shepherd's result implies ours.

## 5. The sharpness of the upper bound

We know of no trees  $T$  of any order  $n$  for which  $\gamma_e(T) = 2(n+2)/5$ . Certainly, there are no trees of order 10 or less that achieve the bound (this may be verified by hand). Whether or not there are larger trees that do remains to be seen. We have conducted a number of computer searches to find trees  $T$  for which the ratio  $\gamma_e(T)/(n+2)$  is as high as possible. At the time of writing, the following two results represent the best we have been able to achieve. We shall use the following notation: For positive integers  $r_0, r_1, \dots, r_d$ , we denote by  $T[r_0, r_1, \dots, r_d]$  the rooted tree in which the root has degree  $r_0$  and every vertex at distance  $i$  from the root has degree  $r_i$ .

**Theorem 8.** *There exists an infinite class  $\mathcal{T}$  of trees such that  $T \in \mathcal{T}$  implies that  $\frac{3}{8} < \frac{\gamma_e(T)}{n+2} \leq \frac{144}{379}$ .*

**Proof.** We provide a sketch of the proof and leave the reader to insert the details where necessary. The trees in the infinite class  $\mathcal{T}$  are parameterized by an integer  $d \geq 2$ . For some such  $d$ ,  $T(d)$  is a rooted tree of depth  $d+4$ ; specifically,  $T(d) = T[4, 4, 3, 3, 3, \dots, 3, 4, 2, 1]$ . We shall call the set of vertices at distance  $a$  from the root,  $r$ , the  $a$ -level. Let  $D$  be the vertices in the  $(d+3)$ -level and let  $S$  be a minimum exponential dominating set of  $T(d)$ . We may assume without loss of generality that  $S \subseteq D$  (vertices of degree 3 or more that are in  $S$  and not in  $D$  serve only to 'block' other dominating vertices). In fact, we claim that  $S = D$ . Suppose, to the contrary that some vertex  $x$  of  $D$  is not in  $S$ . Then, if  $z$  is the endvertex adjacent to  $x$ , we have  $w_S(z) \leq \frac{27}{2^{d+6}} + \frac{7}{8}$ ; since  $d \geq 2$ , this quantity is less than 1, implying that no such vertex  $x$  exists. Finally, it can be verified that  $n = 12 \cdot 2^{d+3} - 7$  and  $\gamma_e(T(d)) = |D| = 36 \cdot 2^d$ .  $\square$

Note that in Theorem 8, the tree  $T$  with the highest ratio  $\gamma_e(T)/(n+2)$  occurs when  $d = 2$ ; here,  $\gamma_e(T)/(n+2) = 144/379 \approx .380$ . While we can find no infinite class of graphs with a higher ratio, we have found graphs with a slightly higher ratio. The following result describes one of them:

**Theorem 9.** *There exists a tree  $T$  of order 375 with  $\gamma_e(T) = 144$ .*

**Proof.** The result follows from considering the tree  $T[2, 3, 3, 4, 3, 4, 2, 1]$ .  $\square$

Note that in Theorem 9,  $\gamma_e(T)/(n+2) = 144/377 \approx .382$ .

## 6. Edge deletion

We briefly consider the effect of edge deletion on the exponential domination number. It is well known that deleting an edge of a graph will either increase the domination number by 1 or leave it unchanged. The situation is different for exponential domination. While removing an edge cannot decrease  $\gamma_e$ , the exponential domination number can increase by an arbitrarily large factor.

To see this, let  $d$  be a large positive integer. Consider the tree  $T = T[r, 3, 3, \dots, 3, 3, 2, 1]$ , where 3 is repeated  $d-2$  times and  $r = 2^{d-1}$ . Then  $T$  has  $2^{2d-3}$  vertices of degree 2. As above, it is easy to verify that the set of vertices of degree 2 forms a  $\gamma_e$ -set of  $T$ , so

$$\gamma_e(T) = 2^{2d-3}.$$

Now consider the graph  $G$  obtained from the disjoint union of  $T$  and the star  $K_{1,t}$ , where  $t = 2^{d+1}$ , by adding an edge  $e$  joining the root of  $T$  and the center vertex of  $K_{1,t}$ . It is easy to check that the set of endvertices of  $K_{1,t}$  is an exponential dominating set of  $G$ , so

$$\gamma_e(G) \leq t = 2^{d+1}.$$

Deleting  $e$  leaves  $T \cup K_{1,t}$ , so

$$\gamma_e(G - e) = \gamma_e(T) + \gamma_e(K_{1,t}) = 2^{2d-3} + 1.$$

Hence

$$\frac{\gamma_e(G - e)}{\gamma_e(G)} = 2^{d-4} + 2^{-d-1},$$

which can be arbitrarily large, provided  $d$  is large enough.

## 7. Open questions

We conclude by listing some questions which we would be interested in knowing the answers to:

1. Under what conditions is  $\gamma_e(G) = \gamma(G)$ ?
2. Let  $T$  be a tree. Is there a polynomial-time algorithm to determine  $\gamma_e(T)$ ?
3. Does there exist a tree (or, even better, an infinite class of trees)  $T$  for which  $144/377 < \frac{\gamma_e(T)}{n+2} \leq \frac{2}{5}$ ?

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